Lecture 10

– Phys IV notes

Review of Lecture 9:

- Matching conditions

$$\vec{E}_2^{\parallel} - \vec{E}_1^{\parallel} = 0 \qquad \vec{D} = \varepsilon \vec{E}$$

$$\vec{H}_2^{\parallel} - \vec{H}_1^{\parallel} = \vec{J}_s \times \vec{n} \qquad \vec{H} = \frac{1}{\mu} \vec{B}$$

$$D_2^{\perp} - D_1^{\perp} = \rho_s$$

$$B_2^{\perp} - B_1^{\perp} = 0$$

$$-\Delta \vec{H} + \varepsilon \mu \ddot{\vec{H}} = 0$$

$$v^2 = \frac{1}{\varepsilon\mu} = c^2 \frac{\varepsilon_0 \mu_0}{\varepsilon \mu} \equiv c^2 \frac{1}{n^2}$$

- Plane waves in medium (see lec. 9)

9.1 EM energy in medium

We would like to derive an expression for the energy density, energy flux and energy conservation in a media with non-trivial μ and ε. We will focus on the isotropic case with constant μ and ε.

Thus, we are looking for the analogs of the vacuum equations:

$$\frac{\partial u}{\partial t} + \vec{E}\cdot\vec{J} + \vec{\nabla}\cdot\vec{S} = 0 \quad , \text{ where}$$

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} \quad \text{[Pointing vector]}$$

$$u = \frac{\varepsilon_0}{2}|\vec{E}|^2 + \frac{1}{2\mu_0}|\vec{B}|^2$$

$$\frac{\partial \varepsilon_{kin}}{\partial t} = \vec{J}\cdot\vec{E} \quad \text{[change in the kinetic energy of the charges]}$$

We derived these equations in Lecture 4 by multiplying second and third Maxwell equations by $\vec{E}$ and $\vec{B}$ correspondingly. We now do the same with the averaged equations and multiply by averaged $\vec{E}$ and $\vec{B}$:

$$\frac{\partial u}{\partial t} + \vec{E} \cdot \underbrace{\langle \vec{j} \rangle}_{\text{the only difference}} + \vec{\nabla} \cdot \vec{S} = 0$$

Now we use the equation for $\langle \vec{j} \rangle$ from Lecture 8:

$$\langle \vec{j} \rangle = \vec{J} + \frac{\partial \vec{P}}{\partial t} + \vec{\nabla} \times \vec{M} \quad , \text{ where}$$

$\vec{E} \cdot \vec{J} \equiv$ Work done on the free charges

$$\frac{\partial u}{\partial t} = \varepsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} + \frac{1}{\mu_0} \vec{B} \cdot \frac{\partial \vec{B}}{\partial t}$$

$$\vec{E}\left(\frac{\varepsilon_0 \partial \vec{E}}{\partial t} + \frac{\partial \vec{P}}{\partial t} \right) = \vec{E} \cdot \varepsilon \frac{\partial \vec{E}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t}\left(\vec{E} \cdot \vec{D} \right)$$

$$\left[\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon \vec{E} \right]$$

Now we need to massage the

$\vec{E} \cdot \vec{\nabla} \times \vec{M}$ term.

We rewrite it as

$$\vec{E} \cdot \vec{\nabla} \times \vec{M} = \underbrace{\vec{\nabla} \cdot \left(\vec{M} \times \vec{E} \right)}_{\text{will get combined with } \vec{\nabla} \cdot \vec{S}} + \vec{M} \left(\vec{\nabla} \times \vec{E} \right) =$$

$$= \vec{\nabla} \cdot \left(\vec{M} \times \vec{E} \right) - \vec{M} \frac{\partial \vec{B}}{\partial t}$$

will combine with $\frac{1}{\mu_0} \vec{B} \cdot \frac{\partial \vec{B}}{\partial t}$

We thus get

$$\frac{\partial u_{eff}}{\partial t} + \vec{\nabla} \cdot \vec{S}_{eff} + \vec{E} \cdot \vec{J} = 0$$

$$u_{eff} = \frac{1}{2} \vec{E} \cdot \vec{D} + \frac{1}{2} \vec{B} \cdot \vec{H}$$

$$\vec{S}_{eff} = \vec{E} \times \vec{H}$$

EM waves in Medium

We again study isotropic linear medium

$$\vec{D} = \varepsilon \vec{E} \qquad \vec{B} = \mu \vec{H}$$

If there are no free charges,

$$\vec{\nabla} \cdot \vec{D} = 0 \qquad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = 0 \qquad \vec{\nabla} \cdot \vec{B} = 0$$

Let's derive the wave equation for the magnetic field (as we did in the first lecture)

$$\vec{\nabla} \times \vec{\nabla} \times \vec{H} - \vec{\nabla} \times \frac{\partial \vec{D}}{\partial t} = 0 \implies$$

$$-\Delta \vec{H} - \vec{\nabla} \times \frac{\partial}{\partial t} \vec{D} = 0$$

$$-\vec{\nabla} \times \frac{\partial}{\partial t} \vec{E} = \frac{\partial^2 \vec{B}}{\partial t^2}$$

$$-\Delta \vec{H} + \varepsilon \mu \frac{\partial^2}{\partial t^2} \vec{H} = 0 \Rightarrow v^2 = \frac{1}{\varepsilon \mu} =$$

$$= c^2 \frac{\varepsilon_0 \mu_0}{\varepsilon \mu} \equiv c^2 \frac{1}{n^2}$$

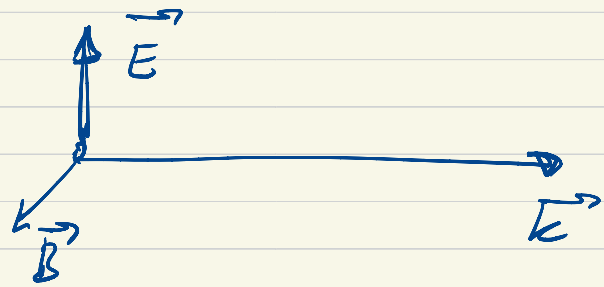
Plane wave $\vec{B} = \vec{B}_0 e^{i\vec{k}\cdot\vec{x} - i\omega t}$ is

a solution, given that

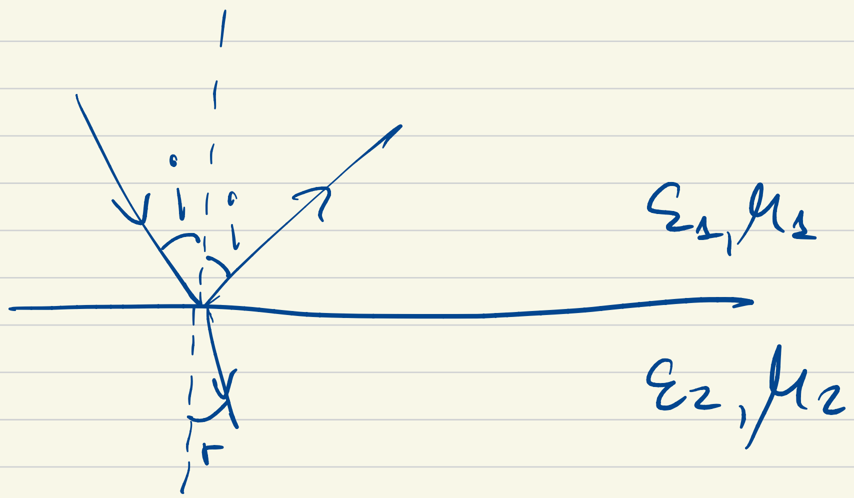
$$\omega = v|\vec{k}|$$

We could repeat a careful derivation we had with scalar and vector potential and conclude that the entire plane wave story goes through:

$$\vec{B} = \vec{k} \times \vec{E} \cdot \frac{1}{\omega}$$



• As an application of matching conditions, one can derive the laws of reflection and transmission through the boundary of two media:



$$\frac{\sin r}{\sin i} = \frac{n_1}{n_2}$$

[Note connection between the ray optics and the wave optics]

Plan of the last 5 lectures

- Galilean invariance and Lorentz invariance
- Physics of Lorentz transformations
- Covariant formulation of EM
- Covariant formulation of Newtonian mechanics
- Action principle for EM

Plan of this lecture

- Historical remarks about Galilei and Lorentz inv.
- Symmetries of Lagrangians → Galilean group and group of rotations
- Failure of the Galilean invariance of the wave equation

- Invariance of the speed of light and wave equation under Lorentz transformations
- Wick rotation and Lorentz transformations as 4D rotations.

Galilean invariance

Galilei 1564–1642 suggested that laws of Physics are invariant under change of the reference frame:

$$\vec{x}' = \vec{x} + \vec{v}t$$

if frame F' moves with respect to F with velocity $\vec{v}$.

This, together with rotations:

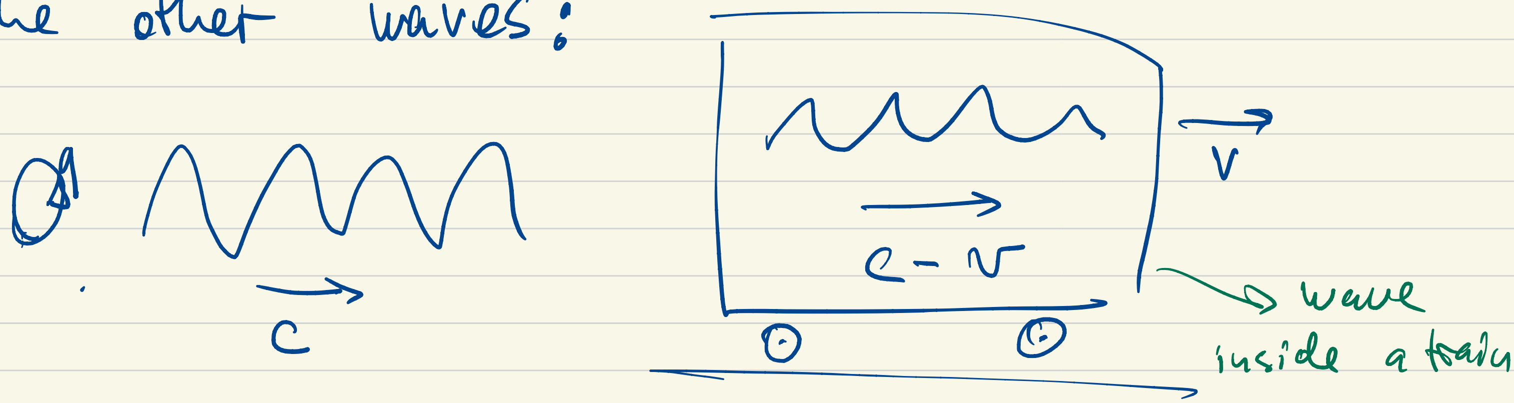
$$\vec{x}' = \hat{R}\circ\vec{x} \equiv \sum_{j=1}^{3} R_{ij}x_j \equiv R_{ij}x_j$$

And translations: $\vec{x}' = \vec{x} + \vec{a}$

$$t' = t + T$$

form the group of symmetries of Newtonian mechanics (Newton 1643 – 1727).

By the end of XIX century it was believed that speed of light has the same properties as that of the other waves:



[light propagates in "aether", hence ε_0, μ_0]

Maxwell eqn's ~ 1870 $\qquad c^2 = \frac{1}{\varepsilon_0 \mu_0}$

However, Michelson and Morley in 1880's showed that speed of light was the same in all frames...

This lead to a puzzle

Constancy of the speed of light ~

~ failure of wave equation (Maxwell eqn's) to be invariant under Galilean transformations.

1900 Kelvin's "Two clouds" lecture

Second "cloud" was failure of classical stat-mech. to explain the radiation spectra.

Theory of relativity

Quantum mechanics

Einstein, 1905

Both in one year!

- Symmetries of Lagrangians → Galilean group and group of rotations

Consider a typical Lagrangian:

$$\mathcal{L} = \sum_i \frac{m_i \dot{\vec{x}}_i^2}{2} - \sum_{i,j} U(|\vec{x}_i - \vec{x}_j|)$$

$$S = \int dt\, \mathcal{L}$$

Equations follow from $\frac{\delta S}{\delta x_i(t)} = 0$

[$\mathcal{L}$ - inv. => EOM inv.]

$$\vec{x}' = \hat{R} \circ \vec{x} \qquad \vec{x}' = \vec{x} + \vec{a}$$

$$t' = t + T$$

invariant ($\hat{R}$ peserves angles and lengths)

Galilean a bit less trivial:

$$\frac{m\dot{\vec{x}}^2}{2} \to \frac{m\dot{\vec{x}}^2}{2} + m\vec{v}\cdot\dot{\vec{x}} + \frac{m}{2}\vec{v}^2$$

$$\mathcal{L} \to \mathcal{L} + \frac{d}{dt}\left(\sum_i m \vec{v} \cdot \vec{x}_i + t \frac{m}{2} \vec{v}^2 \cdot N \right)$$

S is invariant !

$$(\vec{x}, t) \to x^\mu : \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \to \text{4-vector}$$

$$(t', x_1', x_2', x_3') =$$

$$= (t, x_1 + v_1 t, x_2 + v_2 t, x_3 + v_3 t) = \underbrace{\begin{pmatrix} 1 & & & \\ v_1 & 1 & & 0 \\ v_2 & 0 & 1 & \\ v_3 & 0 & 0 & 1 \end{pmatrix}}_{\hat{G}(\vec{v})} \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\hat{G}(\vec{v}) \cdot \hat{G}(\vec{v}\,') = \hat{G}(\vec{v} + \vec{v}\,')$$

Group multiplication

Group of rotations:

$$x_i' = R_{ij} x_j : \quad \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\vec{x}\,' = \hat{R}\,\vec{x}$$

three notations for the same thing.

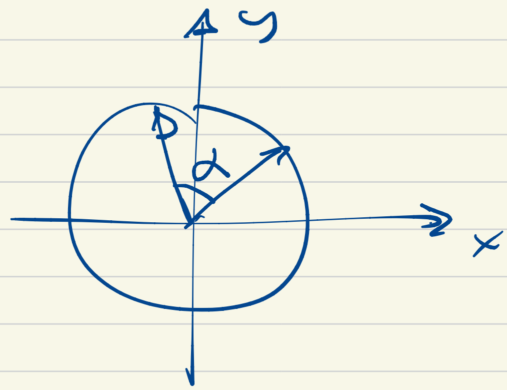
$$x^{\mu\prime} = \hat{R}^4 \cdot x^\mu : \qquad \hat{R}^4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \hat{R} & \\ 0 & & & \end{pmatrix}$$

$$R_{\mu\nu} x_\nu$$

arbitrary element of the rotation group can be written as combination of three rotations:

$$\hat{R} = \hat{R}_{xy} \hat{R}_{yz} \hat{R}_{xz}$$

$$\hat{R}_{xy} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Rotation preserves scalar products:

$$\vec{x}\cdot\vec{y} = \vec{x}\,'\cdot\vec{y}\,' : \qquad x_i' = R_{ij}x_j$$

$$y_i' = R_{ik}\,y_k$$

$$x_i \cdot y_i = x_j R_{ij} \cdot R_{ik} y_k =$$

$$x_j \left(\hat{R}^T \cdot \hat{R}\right)_{jk} y_k \qquad * \qquad R$$

$$x_i \cdot y_i = x_j \hat{I}_{jk} y_k$$

$$\hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \approx \delta_{ij} \text{ (kronecker - delta)}$$

Since $*$ is true for any $\vec{x}$ and $\vec{y}$,

$$\hat{R}^T \cdot \hat{R} = I \Rightarrow \hat{R} \text{ is an orthogonal matrix}$$

$$\det \hat{R}\hat{R}^T = (\det \hat{R})^2 = 1$$

$\hat{R}$ preserves orientation: $\det \hat{R} = 1$

$SO(3)$

$\swarrow$ $\det = 1$ $\quad \hookrightarrow$ orthogonal

Parity:

$$\hat{P} = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

$$\hat{P} \otimes \hat{R} \rightarrow O(3)$$

Failure of the Galilean invariance of the wave equation

$$\Box \psi = 0 \quad \to \quad \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^i} \right) \psi = 0$$

Translations and rotations are OK!

$$\begin{cases} x^{i\prime} = x^i + v^i t \\ t' = t \end{cases}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial x_i'} \cdot \frac{\partial x_i'}{\partial t} + \frac{\partial}{\partial t'} \cdot \frac{\partial t'}{\partial t} =$$

$$= v^i \frac{\partial}{\partial x_i'} + \frac{\partial}{\partial t'}$$

$$\frac{\partial^2}{\partial t^2} \to \frac{\partial^2}{\partial t'^2} + 2 v^i \frac{\partial}{\partial x_i'} \frac{\partial}{\partial t} + \left(v^i \frac{\partial}{\partial x^{i\prime}} \right)^2$$

$$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_i'} \implies$$ wave equation is not invariant!

It is because we did not transform $\underline{\underline{c}}$

$\Rightarrow$ same statement is that constancy of speed of light is in contrast with Galilean invariance:

Summary:

	Translations	Rotations	Galilean
Mechanics	✓	✓	✓
CED	✓	✓	✗

- Wave eqn not invariant $\Rightarrow$ Maxwell eqn. not invariant
- Note: gauge invariance vs <u>global</u> symmetry invariance.

Invariance of the speed of light and wave equation under Lorentz transformations

We would like to suggest that the idea of equivalence of inertial reference frame is still true in EM. However, the transformation laws for coordinates change.

Galilean Boosts $\longrightarrow$ Lorentz Boosts.

Which transformation leaves invariant the wave equation?

$$\left[\frac{1}{c^2}\left(\frac{\partial}{\partial t}\right)^2-\left(\frac{\partial}{\partial x_1}\right)^2-\left(\frac{\partial}{\partial x_2}\right)^2-\left(\frac{\partial}{\partial x_3}\right)^2\right]\psi=0$$

There are several ways to derive it. I prefer the following:

Wick rotation and Lorentz transformations as 4D rotations.

introduce "Euclidean time"

$$x_0^E = -ict \quad : \quad \frac{\partial}{\partial t} = ic \frac{\partial}{\partial x_0^E}$$

this is called "Wick rotation". Then we will rotate back.

the wave equation becomes

$$\underbrace{-\left(\frac{\partial}{\partial x_0^E}\right)^2 \underbrace{-\left(\frac{\partial}{\partial x_1}\right)^2 - \left(\frac{\partial}{\partial x_2}\right)^2 - \left(\frac{\partial}{\partial x_3}\right)^2}_{\text{invariant under } SO(3)} \psi = 0}_{\text{invariant under } SO(4)}$$

invariant under $SO(4)$ – 4D rotations! → absolutely analogous to 3D rotations.

$$\hat{\Lambda}_E \in SO(4): \quad \hat{\Lambda}_E \cdot \hat{\Lambda}_E^T = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

Transformations:

$$x_\mu^E \to x_\mu^{E\prime} = \Lambda^E_{\mu\nu} x^E_\nu \,, \quad y^E_\mu \to y'^E_\mu = \Lambda^E_{\mu\nu} y^E_\nu$$

$$x^E_\mu y^E_\mu = x^E_0 y^E_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 \;; \quad x'^E_\mu \cdot y'^E_\mu =$$

$$= \Lambda_{\mu\rho} x^E_\rho \Lambda^E_{\nu\mu} y^E_\mu = x^E_\rho \underbrace{\left(\Lambda^{E T}\right)_{\rho\nu} \Lambda^E_{\nu\mu}}_{\hat{\Lambda}^{ET} \cdot \hat{\Lambda}^E = \mathbb{I}} y^E_\mu = x_\mu \cdot y_\mu$$

$$\hat{\Lambda}_E = \underbrace{\hat{\Lambda}^E_{01} \hat{\Lambda}^E_{02} \hat{\Lambda}^E_{03}}_{\text{New rotations}} \underbrace{\hat{\Lambda}_{12} \hat{\Lambda}_{13} \hat{\Lambda}_{23}}_{SO(3) \text{ rotations (old),}}$$

they are always Euclidean

$$\hat{\Lambda}^E_{01} : \begin{pmatrix} \cos\alpha & \sin\alpha & \multicolumn{2}{c}{0} \\ -\sin\alpha & \cos\alpha & & \\ \multicolumn{2}{c}{0} & 1 & 0 \\ & & 0 & 1 \end{pmatrix}$$

$$x_0^{E\prime} = x_0 \cos\alpha + x_1 \sin\alpha$$

$$x_1' = x_1 \cos\alpha - x_0 \sin\alpha$$

Now, let us remember the definition of x^E_0: $\quad x^E_0 = -ict \iff t = \dfrac{i x^E_0}{c}$

$$-ict' = -ict\cos\alpha + x_1\sin\alpha$$

$$x_1' = x_1\cos\alpha + ict\sin\alpha$$

$$ct' = ct\cos\alpha + i x_1\sin\alpha$$

$$x_1' = x_1\cos\alpha + ict\sin\alpha$$

define rapidity: $\chi = i\alpha$

$$i\sin\alpha = \sinh\chi \qquad \alpha = -i\chi$$

$$\cos\alpha = \cosh\chi$$

$$\left.\begin{aligned} t' &= t\cosh\chi + \frac{x_1}{c}\sinh\chi \\ x_1' &= x_1\cosh\chi + ct\sinh\chi \end{aligned}\right\} \text{Lorentz Boost}$$

$\beta \to 0$: $\beta \sim \frac{v_1}{c}$

$$\left.\begin{aligned} t' &= t + O\left(\frac{1}{c^2}\right) \\ x_1' &= x_1 + vt + O\left(\frac{1}{c^2}\right) \end{aligned}\right\} \text{Galilean Boosts}$$

If Euclidean objects are invariant under $SO(4)$ rotations Lorentzian objects defined through $t = \frac{i x_0^E}{c}$ will be invariant under Lorentz transformations defined through $\chi = i\alpha$ (they form $SO(1,3)$ group).

Now we define actual Lorentz transformations:

$$\hat{\Lambda} = \hat{\Lambda}_{01} \hat{\Lambda}_{02} \hat{\Lambda}_{03} \hat{\Lambda}_{12} \hat{\Lambda}_{13} \hat{\Lambda}_{23}$$

$$\hat{\Lambda}_{01} = \begin{pmatrix} \cosh\chi & \sinh\chi & 0 & 0 \\ \sinh\chi & \cosh\chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

same for other boosts.